ON THE RELATION BETWEEN THE THREE-PARAMETER GROUPS OF A CUBIC SPACE CURVE AND A QUADRIC SURFACE*

BY

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§ 1. *Statement of the problem.*‡

As is well known, there is a three-parameter group, G_3, of projective transformations which leaves unaltered a cubic curve, C^3, in a space of three dimensions, S_3. The group, F_3, of algebraic transformations, reciprocal to G_3 also leaves C^3 unaltered.

The six-parameter projective group which leaves a quadric, Q, in a three dimensional space, Σ_3, unaltered contains two three-parameter subgroups, Γ_3 and Φ_3, each of which is defined by its leaving unaltered every one of a set of generators of Q.

That the groups G_3 and Γ_3 are similar LIE has pointed out. He has given also a transformation which carries the one group into the other. But the form of this transformation is not such as to permit of an easy discussion of its properties. It is the object of this paper to set forth a transformation, T, which carries G_3 into Γ_3, in such a form that its effect upon the various manifolds in S_3 and Σ_3 may be more easily studied. This object will be effected by first obtaining the integral equations of G_3 and Γ_3 in readily comparable forms. Possibly the chief interest of the method lies in the fact that the algebraic transformation T will also transform the projective group Φ_3 into the algebraic group F_3. Properties of F_3 may then through the knowledge of T be inferred from those of Φ_3.

§ 2. *The trilinear binary form.*

The general trilinear binary form, written symbolically as

$$A \equiv (\alpha x)\beta y)(\gamma z),$$

involves homogeneously eight constants—its system of coefficients. If these coefficients or properly selected linear combinations of them be considered as

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coördinates in a linear seven-dimensional space, S_7, we obtain a one-to-one correspondence between the points of S_7 and the totality of forms A. Expanding A according to the CLEBSCH-GORDAN formula, we have

$$A = A_1 + A_2,$$

where

$$A_1 = \tfrac{1}{3}\left[(\alpha x)(\beta z)(\gamma y) + (\alpha z)(\beta y)(\gamma x) + (\alpha y)(\beta x)(\gamma z)\right],$$

$$A_2 = \tfrac{1}{3}\left[(yz)(\beta\gamma)(\alpha x) + (zx)(\gamma\alpha)(\beta y) + (xy)(\alpha\beta)(\gamma z)\right].$$

If

$$(px)^3 \equiv (\alpha x)(\beta x)(\gamma x) \qquad \text{then} \qquad A_1 = (px)(py)(pz).$$

Hence A_1 and A_2 each depend homogeneously upon four constants: the former upon the coefficients of the cubic $(px)^3$; the latter upon the six quantities $(\beta\gamma)\alpha_\iota$, $(\gamma\alpha)\beta_\iota$, and $(\alpha\beta)\gamma_\iota$ $(\iota = 1, 2)$, among which exist the two linear relations given by the identical vanishing of

$$(\beta\gamma)(\alpha x) + (\gamma\alpha)(\beta x) + (\alpha\beta)(\gamma x).$$

In the aggregate of forms A occur two special linear aggregates: that of the forms A_1 represented by the points of a three-dimensional spread, S_3, in S_7; and that of the forms A_2 represented by the points of a three-dimensional spread, Σ_3, in S_7. Since A_1 and A_2 do not vanish simultaneously, S_3 and Σ_3 are *skew* spaces. If the forms A (x, y, and z considered cogredient) be transformed by the general binary projective group, the space S_7 is transformed by a three-parameter projective group, which, in the *invariant space* S_3, leaves a cubic space curve unaltered and, in the *invariant space* Σ_3, leaves a quadric unaltered —the quadric

$$(\beta\gamma)(\alpha\beta')(\gamma'\alpha') = (\gamma\alpha)(\beta\gamma')(\alpha'\beta') = (\alpha\beta)(\gamma\alpha')(\beta'\gamma') = 0.$$

The trilinear form has now served its purpose in having suggested the following coördinate systems in S_3 and Σ_3. In S_3 we take as the coördinates of a point the coefficients of a binary cubic form, $(px)^3$. In Σ_3 we take as the coördinates of a point the six quantities l_ι, m_ι, n_ι $(\iota = 1, 2)$, connected by the identity

$$(lx) + (mx) + (nx) = 0.$$

§ 3. *The group G_3 and its invariant systems of manifolds in S_3.*

The representation of points in an S_3 by binary cubic forms is well known. For our present purposes we use the following notation for the comitants of the cubic and resolution of the cubic into its linear factors given by E. STUDY.* We take

$$f = p = (px)^3 = (p'x)^3, \qquad \delta = (\delta x)^2 = \tfrac{1}{2}(pp')^2(px)(p'x),$$

* American Journal of Mathematics, vol. 17 (1895), p. 187.

$$q = (qx)^3 = 2(p\,,\,\delta) = (pp')^2(p''p)(p'x)(p''x)^2,$$
$$r = \tfrac{1}{2}(p\,,\,q)^3 = 2(\delta\delta')^2 = \tfrac{1}{2}(pp')^2(p''p)(p'''p')(p'''p'')^2.$$

The syzygy between these forms is

$$4\delta^3 + q^2 + rf^2 = 0\,;$$

whence

$$-\,4\delta^3 = \{q + \sqrt{-r}f\}\{q - \sqrt{-r}f\}.$$

The linear factors (σx) and (τx) of δ are defined as

$$(\sigma x) = \sqrt[3]{\frac{q + \sqrt{-r}f}{2}}, \qquad (\tau x) = \sqrt[3]{\frac{q - \sqrt{-r}f}{2}}, \qquad (\sigma x)(\tau x) = -\,\delta.$$

Hence

$$(\sigma x)^3 + (\tau x)^3 = q, \qquad (\sigma x)^3 - (\tau x)^3 = \sqrt{-r}f,$$
$$(\sigma\tau)^3 = r\sqrt{-r}, \qquad (\sigma\tau)^2 = -\,r, \qquad \therefore (\sigma\tau) = -\sqrt{-r}.$$

If ϵ is an imaginary cube root of unity and ϵ_1, ϵ_2, ϵ_3 a cyclical permutation of 1, ϵ, ϵ^2, also if $\bar{\epsilon}_\iota$ is the conjugate of ϵ_ι, three linear forms (λx), (μx), (νx) are defined by the equations

$$(\sigma\tau)(\lambda x) = \bar{\epsilon}_1(\sigma x) - \epsilon_1(\tau x), \qquad (\sigma\tau)(\mu x) = \bar{\epsilon}_2(\sigma x) - \epsilon_2(\tau x),$$
$$(\sigma\tau)(\nu x) = \bar{\epsilon}_3(\sigma x) - \epsilon_3(\tau x)\,;$$

and it follows further that

$$0 = (\lambda x) + (\mu x) + (\nu x),$$
$$3\delta = r\{(\mu x)(\nu x) + (\nu x)(\lambda x) + (\lambda x)(\mu x)\} = -\frac{r}{2}\sum(\lambda x)^2,$$
$$p = r(\lambda x)(\mu x)(\nu x) = \frac{r}{3}\sum(\lambda x)^3,$$
$$3\sqrt{-3}\,q = \{(\mu x) - (\nu x)\}\{(\nu x) - (\lambda x)\}\{(\lambda x) - (\mu x)\},$$
$$\frac{-\sqrt{-3}}{(\sigma\tau)} = \frac{\sqrt{-3}}{\sqrt{-r}} = (\mu\nu) = (\nu\lambda) = (\lambda\mu).$$

We can now write the group G_3 in the form

$$(A) \quad (\lambda' x) = (\lambda d)(\delta x), \qquad (\mu' x) = (\mu d)(\delta x), \qquad (\nu' x) = (\nu d)(\delta x),$$

where $(dy)(\delta x)$ is a general linear transformation in the binary domain. Any one of these three identities in x is a consequence of the other two. Since the forms (λx), (μx) and (νx) are defined on the supposition that $r \neq 0$, the group G_3 in this form is defined only for points in general position. This is sufficient however to completely determine the group.

Hereafter we consider $(px)^3$ a variable point (undetermined cubic form) and retain for functions of its coördinates (coefficients) the above notation. The forms $(p_1x)^3$, $(p_2x)^3$, etc., denote fixed points and their comitants are distinguished from those of $(px)^3$ by the use of the respective suffix. A point on C^3 is given simply by a linear form. Then we can state that

(1) $(pp_1)^3 = 0$ *is a plane cutting* C^3 *in the points* $(\lambda_1 x)$, $(\mu_1 x)$, $(\nu_1 x)$ *and passing through* $(p_1x)^3$.

(2) $r = 0$ *is the ruled surface of tangents of* C^3 *and contains* C^3 *as a cuspidal edge. Or it is an algebraic surface of the fourth order containing* C^3 *as a double doubly asymptotic curve.*

Here we understand by an m-tuple p-tuply asymptotic curve on a surface, a curve of m-tuple points such that the tangent cone of order m at every point on the curve contains the osculating plane of the curve at that point p times.

(3) $(qp_1)^3 = 0$ *is the polar cubic surface of* $(p_1x)^3$ *as to* $r = 0$. *It contains* C^3 *as an asymptotic curve but has double points at* $(\lambda_1 x)$, $(\mu_1 x)$, $(\nu_1 x)$. *It meets* $r = 0$ *in the three tangents to* C^3 *at these double points and in* C^3 *taken three times.*

(4) $(\delta a)^2 = 0$, *where* $(ax)^2 = (a_1x)\cdot(a_2x)$ *is a general quadratic binary form, is the most general quadric containing* C^3. *The system of generators which are chords (double secants) of* C^3 *meets* C^3 *in pairs of points apolar to* $(ax)^2 = 0$. *The two tangents of* C^3 *at* (a_1x) *and* (a_2x) *together with* C^3 *taken twice form the intersection of* $(\delta a)^2 = 0$ *and* $r = 0$. *If* $(ax)^2$ *has a double factor* (a_1x), *the quadric is the cone containing* C^3 *with vertex at* (a_1x).

If now, for brevity, we write $[(p\alpha)^3]^n$ for $(p\alpha)^3(p'\alpha)^3\cdots(p^{(n-1)}\alpha)^3$ and use corresponding abbreviations for the other concomitants of $(px)^3$ we can state the theorem:

(5)* *The most general algebraic surface of order* n *in* S_3 *can be written*

$$\Sigma\,[(p\alpha)^3]^{n_1}[(\delta\alpha)^2]^{n_2}[(q\alpha)^3]^{n_3}r^{n_4} = 0,$$

where $n^3 \leqq 1$ *and* $(\alpha x)^{3n_1+2n_2+3n_3}$ *is a general binary form of that order, the summation being extended over all positive integer solutions of* $n_1 + 2n_2 + 3n_3 + 4n_4 = n$.

(6) *The most general algebraic surface of order n containing* C^3 *as an m-tuple p-tuply asymptotic curve is that of* (5) *where the exponents satisfy the further conditions*

$$n_2 + n_3 + 2n_4 \leqq m, \qquad n_3 + 2n_4 \leqq p \qquad (p \leqq m)$$

and the summation contains a term satisfying both equalities.

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*Of the above (1), (2), (3) and (4) are well-known manifolds connected with C^3. (5) and (6) are proved in an article by the author to appear later. The proof of (7) rests simply on the application of Aronhold's process to the comitants of the cubic. The sextic in (7) in conformity with the requirements $n_3 \leqq 1$ can be written

$$r_1\cdot(\delta p_1)^2(\delta' p_1')^2(\delta'' p_1)(\delta'' p_1') - r\cdot(\delta_1 p)^2(\delta_1' p')^2(\delta_1'' p)(\delta_1'' p') = 0.$$

A particular surface which turns up later is

$$S^6(p_1) = r\,[(pq_1)^3]^2 - r_1\,[(p_1q)^3]^2 = 0.$$

Of this the following properties are easily verified:

(7) *The sextic surface $S^6(p_1)$ contains C^3 as a double doubly asymptotic curve with triple points at $(\sigma_1 x)$ and $(\tau_1 x)$. At the point $(p_1x)^3$ it has a triple point. The tangent cone at the triple point osculates C^3 at $(\sigma_1 x)$ and $(\tau_1 x)$ and cuts it at the points given by $(q_1x) = 0$. The surface contains the line $(p_1x)^3 + \rho(q_1x)^3$ as a double line (except for the three triple points). The tangent cone at $(q_1x)^3$ is $(q_1p)^3 = 0$ taken twice.*

The system of surfaces $S^6(p_1)$ is transformed into itself by G_3 and the group is six-tuply transitive with regard to general members of the system.

§ 4. *The quadric Q in Σ_3 and the groups Γ_3 and Φ_3.*

A point in Σ_3 being given by the coefficients of the three binary forms (lx), (mx), (nx) for which always the identity

$$(1)\qquad (lx) + (mx) + (nx) = 0$$

holds, and therefore

$$(2)\qquad (mn) = (nl) = (lm),$$

we have, as the equation of a quadric Q,

$$(3)\qquad (mn) = (nl) = (lm) = 0.$$

A system of generators, say the h-generators, of Q is given by the identity

$$(4)\qquad \rho(lx) + \sigma(mx) + \tau(nx) = 0 \qquad (\rho:\sigma:\tau \neq 1:1:1)$$

By the use of (1), identity (4) may be written in infinitely many forms but we shall take *usually* that one for which $\rho + \sigma + \tau = 0$. The h-generators are then determined by a binary value system ρ, σ, τ.

Three of these generators, denoted hereafter by a, b, and c respectively and given analytically by

$$(5)\qquad \begin{aligned} -3(lx) &= -2(lx) + (mx) + (nx) = 0,\\ -3(mx) &= (lx) - 2(mx) + (nx) = 0,\\ -3(nx) &= (lx) + (mx) - 2(nx) = 0,\end{aligned}$$

will be called the "principal generators." The Hessian pair $H(a, b, c)$ of these three are

$$(6)\qquad (lx) + \omega(mx) + \omega^2(nx) = 0, \qquad (lx) + \omega^2(mx) + \omega(nx) = 0,$$

where ω is an imaginary cube root of unity. The generators forming the cubic covariant of the principal generators, denoted respectively by a', b', c', are

$$(7)\qquad (mx) - (nx) = 0, \qquad (nx) - (lx) = 0, \qquad (lx) - (mx) = 0.$$

The second system of generators, say the κ-generators, of Q is defined by the simultaneous holding of

(8) $$(lu)=0, \qquad (mu)=0, \qquad (nu)=0,$$

in which $u_1:u_2$ is an arbitrary but fixed value. Any one equation of (8) is a consequence of the other two. The κ-system is thus also determined by a binary value system $u_1:u_2$. The plane

(9) $$\rho(lu)+\sigma(mu)+\tau(nu)=0$$

is a tangent plane of Q containing the h-generator (ρ, σ, τ) and the κ-generator (u). Then the tangent planes containing a and the κ-generator (l'); b and the κ-generator (m'); c and the κ-generator (n') are respectively $(ll')=0$, $(mm')=0$ and $(nn')=0$ and they meet in the point whose coördinates are $(l'x)$, $(m'x)$, $(n'x)$. Thus if the principal generators are fixed as well as a binary value system on any one of them then our coördinate system is fixed. If the l', m', n' are permuted in all possible ways six points are obtained corresponding to the six possible ways of coördinating the three κ-generators with a, b, c. Such a set of points will be called a 6-point and be said to be defined by a 3-κ (l', m', n'). By very simple analysis we verify that in general

(10) *A* 6-*point* (l, m, n) *lies on two lines which form the diagonals of the skew quadrilateral on* Q *whose sides are* $H(a, b, c)$ *and the Hessian pair of the* 3-κ (l, m, n). *These two lines are conjugate lines of* Q *and each intersects* Q *in the Hessian pair of the three points on it.*

The construction of a 6-point just given is not valid if $(mn)=(nl)=(lm)=0$. The point then has coördinates $\rho_1(ux)$, $\sigma_1(ux)$, $\tau_1(ux)$ where $\rho_1+\sigma_1+\tau_1=0$ and lies on a κ-generator (u) and an h-generator (ρ, σ, τ) where $\rho+\sigma+\tau=0$ and $\rho\rho_1+\sigma\sigma_1+\tau\tau_1=0$. In general the three quantities, ρ, σ, τ are distinct and different from zero and the 6-point is the six intersections of the κ-generator (u) with the six h-generators obtained by permuting ρ, σ, τ. But if $\rho:\sigma:\tau=-2:1:1$, the 6-point is the three meets of a, b, c with the κ-generator u; if $\rho:\sigma:\tau=0:1:-1$ the 6-point is the three meets of a', b', c' with $\kappa(u)$; while if $\rho:\sigma:\tau=1:\omega:\omega^2$ the 6-point is the two meets of $H(a, b, c)$ with $\kappa(u)$.

The equation of a plane in Σ_3 may always, by the use of (1), be put in the form

(11) $$(l\bar{l})+(m\bar{m})+(n\bar{n})=0$$

so that the coefficients or plane coördinates $(\bar{l}x)$, $(\bar{m}x)$, $(\bar{n}x)$ also satisfy the identity (1). Hence as for points we have 6-planes whose coördinates are the permutations of the coördinates of any one of the six.

(12) *Each plane of the* 6-*plane* (l, m, n) *passes through a line of the* 6-*point* (l, m, n) *and a point on the other line. This relation of the two is reciprocal.*

The construction of the 6-*plane is entirely dual to that of the* 6-*point except that either the* a', b', c' *take the place of* a, b, c *or the cubic covariant of the* 3-κ (l, m, n) *takes the place of the* 3-κ *itself.*

The polar planes as to Q *of a* 6-*point* (l, m, n) *form a* 6-*plane* $(m - n, n - l, l \quad m)$ *whose construction is entirely dual to that of the* 6-*point.*

The identities in x

$$(13) \quad (l'x) = (dl)(\delta x), \qquad (m'x) = (dm)(\delta x), \qquad (n'x) = (dn)(\delta x),$$

in which, as before, $(dy)(\delta x) = 0$ is the general linear transformation in the binary domain, represent a transformation of the point, l, m, n into the point l', m', n' which leaves unaltered both the identities (1) and (4) and, therefore, every generator h. Hence

The identities (13) *are the equations of* Γ_3.

And further the identities

$$(14) \quad \begin{aligned} (l'x) &= a_1(lx) + a_2(mx) + a_3(nx), \\ (m'x) &= b_1(lx) + b_2(mx) + b_3(nx), \\ (n'x) &= c_1(lx) + c_2(mx) + c_3(nx), \end{aligned}$$

in which $(a_1 + b_1 + c_1) : (a_2 + b_2 + c_2) : (a_3 + b_3 + c_3) = 1 : 1 : 1$, leave unaltered the identity (1) and the equations (8) and, therefore, all κ-generators. By the use of (1) these may be written more compactly

$$(15) \quad \begin{aligned} (l'x) &= a_2(mx) + a_3(nx), \\ (m'x) &= b_3(nx) + b_1(lx), \\ (n'x) &= c_1(lx) + c_2(mx), \end{aligned}$$

in which $(b_1 + c_1) : (c_2 + a_2) : (a_3 + b_3) = 1 : 1 : 1$. Hence

The identities (15) *are the equations of* Φ_3.

From the form of (13) and (15) we see that the order of succession is immaterial, i. e., *the group* Φ_3 *is the group reciprocal to* Γ_3. A simple transformation that carries the one group into the other is the harmonic perspectivity with center of perspection the point l'', m'', n'' (not on Q) and plane of perspection the polar plane of this point as to Q. This transformation, S, reads

$$(16) \quad \begin{aligned} (m''n'')\cdot(l'x) &= [(mn'') + (m''n)](l''x) - (m''n'')\cdot(lx), \\ (n''l'')\cdot(m'x) &= [(nl'') + (n''l)](m''x) - (n''l'')\cdot(mx), \\ (l''m'')\cdot(n'x) &= [(lm'') + (l''m)](n''x) - (l''m'')\cdot(nx). \end{aligned}$$

And the transform of (13) by (16) reduces to

$$
\begin{aligned}
&-(m''n'')\cdot(l'x) = (dn'')(\delta l'')\cdot(mx) - (dm'')(\delta l'')\cdot(nx),\\
(17)\qquad &-(n''l'')\cdot(m'x) = (dl'')(\delta m'')\cdot(nx) - (dn'')(\delta m'')\cdot(lx),\\
&-(l''m'')\cdot(n'x) = (dm'')(\delta n'')\cdot(lx) - (dl'')(\delta n'')\cdot(mx).
\end{aligned}
$$

This is the group (15) or Φ_3.

It contains a finite group, g_6, of six transformations which gives rise to the six permutations of l, m, n, i. e., it is the group which leaves every 6-point and every 6-plane unaltered.

The group Φ_3 also contains the special transformation, D, given by

$$(18)\qquad (l'x) = (mx) - (nx), \quad (m'x) = (nx) - (lx), \quad (n'x) = (lx) - (mx).$$

D is "interchangeable" with any transformation of g_6; and D and the transformations of g_6 are the only transformations of Φ_3 which carry 6-points into 6-points.

§ 5. *The transformation T.*

A comparison of the integral equations of G_3 and Γ obtained above suggests at once the transformation, T, which carries the one group into the other. Introducing for convenience later a factor of proportionality, we will define T by means of the identities

$$(1)\qquad (\lambda x) = \frac{3}{(mn)^2}(lx), \qquad (\mu x) = \frac{3}{(nl)^2}(mx), \qquad (\nu x) = \frac{3}{(lm)^2}(nx),$$

viewing this as a transformation of the space S_3 into the space Σ_3. The form is so simple however that we may also consider (1) as T^{-1}, the inverse of T which transforms the space Σ_3 into the space S_3.

Since (λx), (μx), (νx) are defined only to within a permutation we have

(2) *T is an algebraic transformation of the space S_3 into Σ_3, one point of S_3 being transformed into a 6-point of Σ_3. By T^{-1} one point of Σ_3 is transformed into one point of S_3. T transforms the six-tuply transitive group G_3 into the simply transitive group Γ_3.*

By a well known theorem, T^{-1} will then transform the group Φ_3, reciprocal to Γ^3, into the group F_3, reciprocal to G_3 whence from (15), § 4, we have *the identities*

$$
\begin{aligned}
&(\lambda' x) = a_2(\mu x) + a_3(\nu x),\\
(3)\qquad &(\mu' x) = b_3(\nu x) + b_1(\lambda x),\\
&(\nu' x) = c_1(\lambda x) + c_2(\mu x),
\end{aligned}
$$

in which $(b_1 + c_1) : (c_2 + a_2) : (a_3 + b_3) = 1 : 1 : 1$ are the equations of the algebraic group F_3.

The translation of the property of S, (16), § 4, gives

(4) *The transformation S in which $l^{(\iota)}$, $m^{(\iota)}$, $n^{(\iota)}$ are replaced by $\lambda^{(\iota)}$, $\mu^{(\iota)}$, $\nu^{(\iota)}$ carries the projective group G_3 into its reciprocal algebraic group F^3.*

Having now obtained the various groups and the transformations S and T in the desired form there remains the study of the effect of these transformations upon certain manifolds and the resulting derivation of some properties of the groups.

§ 6. *The transforms by T of manifolds in S_3.*

The cubic $(px)^3$ may be written as $r(\lambda x)(\mu x)(\nu x)$ or $3(\lambda x)(\mu x)(\nu x)/(\lambda\mu)^2$. From the formulæ (1) § 5, $(\lambda\mu) = 9/(lm)^3$. Hence, by T, $(px)^3 = (lx)(mx)(nx)$ and the plane $(pp_1)^3 = 0$ becomes $(lp_1)(mp_1)(np_1) = 0$, a cubic surface. Using the comitants of $(p_1x)^3$, this surface may be written in various ways and its properties easily deduced. Thus

$$(1)\quad (lp_1)(mp_1)(np_1) = \tfrac{1}{3}\left[(lp_1)^3 + (mp_1)^3 + (np_1)^3\right]$$

$$= \frac{1}{3\sqrt{-r_1}}\left[(l\sigma_1)^3 + (m\sigma_1)^3 + (n\sigma_1)^3 - (l\tau_1)^3 - (m\tau_1)^3 - (n\tau_1)^3\right]$$

$$(2)\quad = \frac{1}{\sqrt{-r_1}}\left[(l\sigma_1)(m\sigma_1)(n\sigma_1) - (l\tau_1)(m\tau_1)(n\tau_1)\right]$$

$$= \frac{r_1}{9}\left[(l\lambda_1)^3+(m\lambda_1)^3+(n\lambda_1)^3+(l\mu_1)^3+(m\mu_1)^3+(n\mu_1)^3+(l\nu_1)^3+(m\nu_1)^3+(n\nu_1)^3\right]$$

$$= \frac{r_1}{3}\left[(l\lambda_1)(l\mu_1)(l\nu_1) + (m\lambda_1)(m\mu_1)(m\nu_1) + (n\lambda_1)(n\mu_1)(n\nu_1)\right]$$

$$(3) = \frac{r_1}{27}\{[(l\lambda_1)+(m\mu_1)+(n\nu_1)][(l\mu_1)+(m\nu_1)+(n\lambda_1)][(l\nu_1)+(m\lambda_1)+(n\mu_1)]$$
$$+ [(l\lambda_1)+(m\nu_1)+(n\mu_1)][(l\nu_1)+(m\mu_1)+(n\lambda_1)][(l\mu_1)+(m\lambda_1)+(n\nu_1)]\}$$

$$(4) = \frac{r_1}{27}\{[(l\lambda_1) + \omega(m\lambda_1) + \omega^2(n\lambda_1)][(l\mu_1) + \omega(m\mu_1) + \omega^2(n\mu_1)][(l\nu_1)$$
$$+ \omega(m\nu_1) + \omega^2(n\nu_1)] + [(l\lambda_1) + \omega^2(m\lambda_1) + \omega(n\lambda_1)][(l\mu_1)$$
$$+ \omega^2(m\mu_1) + \omega(n\mu_1)][(l\nu_1) + \omega^2(m\nu_1) + \omega(n\nu_1)]\}$$

$$(5) = \frac{1}{27\sqrt{-r_1}}\{[(l\sigma_1) + \omega(m\sigma_1) + \omega^2(n\sigma_1)]^3 + [(l\sigma_1) + \omega^2(m\sigma_1)+\omega(n\sigma_1)]^3$$
$$- [(l\tau_1) + \omega(m\tau_1) + \omega^2(n\tau_1)]^3 - [(l\tau_1) + \omega^2(m\tau_1) + \omega(n\tau)]^3\}.$$

Calling now the six points in which the 3-$\kappa(\lambda_1, \mu_1, \nu_1)$ meets the two generators $H(a, b, c)$ or h_1 and h_2 respectively h_{11}, h_{21}; h_{12}, h_{22}; h_{13}, h_{23}; and further the six points in which the Hessian generators, $(\sigma_1 x)$ and $(\tau_1 x)$, of the 3-κ meet a, b, c respectively σ_1, τ_1; σ_2, τ_2; σ_3, τ_3; and recalling that the 6-point $(\lambda_1, \mu_1, \nu_1)$ is made up of two sets of three points I_1, I_2, I_3 and J_1, J_2, J_3 lying respectively on two lines L_1 and L_2, we may with reference to (5) state:

(6) *The plane* $(pp_1)^3 = 0$ *in* S_3 *is transformed by* T *into the tetrahedral cubic surface* (1). *The vertices of the tetrahedron are the four meets of* $H(a, b, c)$ *and the Hessian* κ*-generators* $(\sigma_1 x)$ *and* $(\tau_1 x)$. *The planes of the tetrahedron are the tangent planes of* Q *at the vertices, i. e., the tetrahedron is inscribed in and circumscribed to* Q. *The third pair of opposite edges is* L_1 *and* L_2.

From (2), (3) and (4) we may read off the situation of the right lines of the surface.

(7) *The* 27 *straight lines on the cubic surface* (1) *are* $\overline{h_{1\iota} h_{2\kappa}}$, $\overline{\sigma_\iota \tau_\kappa}$ *and* $\overline{I_\iota J_\kappa}$ $(\iota, \kappa = 1, 2, 3)$. *The surface cuts* Q *in the three h-generators* a, b, c *and the three* κ*-generators* $(\lambda_1 x)$, $(\mu_1 x)$, $(\nu_1 x)$.

If $(p_1 x)^3$ has a double factor, say $(p_1 x)^3 = (ax)^2 (bx)$ where (ax) and (bx) are linear forms, $(pp_1)^3 = 0$ is a tangent plane of C^3 at (ax). The cubic surface (1) is now

$$
\begin{aligned}
(lp_1)(mp_1)(np_1) &= \tfrac{1}{3}\left[(la)^2(lb) + (ma)^2(mb) + (na)^2(nb)\right] \\
&= \tfrac{1}{27}\left\{\left[(la) + \omega(ma) + \omega^2(na)\right]^2\left[(lb) + \omega(mb) + \omega^2(nb)\right]\right. \\
&\quad \left. + \left[(la) + \omega^2(ma) + \omega(na)\right]^2\left[(lb) + \omega^2(mb) + \omega(nb)\right]\right\}.
\end{aligned} \tag{8}
$$

Hence, calling the generators κ given by (ax) and (bx) the double and single generators respectively, we have

(9) *The cubic surface in* Σ_3 *corresponding by* T *to a tangent plane of* C^3 *in* S_3 *has the double generator for a double line. It is a ruled surface whose lines run across the double and single generators, two through every point of the first and one through every point of the second. Through the points where the double generator meets* a, b, c *run the lines to the points where the single generators meet* a, b, c *and* a', b', c'.

If finally $(p_1 x)^3$ has a triple factor (ax), the surface (1) is $(la)\cdot(ma)\cdot(na)$, i. e., three planes. Hence

(10) *The cubic surface is* Σ_3 *corresponding by* T *to the osculating plane of* C^3 *at* (ax) *in* S_3 *is the three tangent planes of* Q *at the points where the* κ*-generator* (ax) *meets* a, b, c.

In general

(11) *To the triply infinite linear system of planes in* S_3 *corresponds by* T *the triply infinite linear system of tetrahedral cubic surfaces in* Σ_3 *having for common lines the three principal generators* a, b, c.

Three general surfaces of this system having in common a curve of degree 3 and class 0 meet further in six points—the 6-point corresponding to the meet of the three planes in S_3.

We take up now the cubic surfaces in S_3 defined by $(qp_1)^3 = 0$. Since

$$
(qx)^3 = \frac{1}{(\mu\nu)^3}\{(\mu x) - (\nu x)\}\{(\nu x) - (\lambda x)\}\{(\lambda x) - (\mu x)\}
$$

we have on applying T

$$(qx)^3 = \frac{(mn)^3}{27}\{(mx)-(nx)\}\{(nx)-(lx)\}\{(lx)-(mx)\};$$

hence

$$(12)\quad (qp_1)^3 = \frac{(mn)^3}{27}\{(mp_1)-(np_1)\}\{(np_1)-(lp_1)\}\{(lp_1)-(mp)\}.$$

The transformation, D, [(18), § 4] which interchanges a, b, c with a', b', c', changes (12) into

$$\frac{(m'n')^3}{27}(l'p_1)(m'p_1)(n'p_1),$$

which is of the same type as (1). Hence

(13) *To the cubic surface* $(qp_1)^3 = 0$ *in* S_3 *corresponds by* T *in* Σ_3, *besides the quadric* Q *counting three times, a tetrahedral cubic surface, the transform of* (1) *by the harmonic axial collineation,* D, *whose axes are* $H(a, b, c)$.

In the case of the quadric through C^3, $(\delta a)^2 = 0$, we write

$$(\delta x)^2 = -\frac{r}{6}\Sigma(\lambda x)^2 = -\frac{3}{(\mu\nu)^2}\Sigma(\lambda x)^2,$$

or

$$(\delta x)^2 = -\frac{3}{(\mu\nu)^2}\{(\lambda x)+\omega(\mu x)+\omega^2(\nu x)\}\{(\lambda x)+\omega^2(\mu x)+\omega(\nu x)\}$$

$$= -\frac{1}{27}(mn)^2\{(lx)+\omega(mx)+\omega^2(nx)\}\{(lx)+\omega^2(mx)+\omega(nx)\}.$$

If $(ax)^2 = (a_1x)\cdot(a_2x)$ and $(bx)^2 = (b_1x)\cdot(b_2x)$ is apolar to $(ax)^2$ we may write

$$(14)\quad \begin{aligned}(\delta a)^2 = -\frac{1}{54}(mn)^2\{&[(lb_1)+\omega(mb_1)+\omega^2(nb_1)][(lb_1)+\omega^2(mb_1)+\omega(nb_1)\\ &+[(lb_2)+\omega(mb_2)+\omega^2(nb_2)][(lb_1)+\omega^2(mb_1)+\omega(nb_1)]\}.\end{aligned}$$

Hence we have

(15) *To the quadric* $(\delta a)^2 = 0$ *in* S_3 *corresponds by* T *in* Σ_3, *besides* Q *counting twice, a quadric which intersects* Q *in* $H(a, b, c)$ *and the two* κ*-generators* (a_1x) *and* (a_2x). *The two are included in a set of generators* $\bar{\kappa}$ *of* (15) *obtained by taking the two diagonals of all quadrilaterals formed with* $H(a, b, c)$ *by a pair of* κ*-generators apolar to* $(ax)^2$. *Each diagonal pair of generators* $\bar{\kappa}$ *corresponds to one of the set of generators of* $(\delta a)^2 = 0$ *which are chords of* C^3. *

If $(ax)^2$ *has a repeated factor* (a_1x), $(\delta a)^2 = 0$ *becomes the two tangent planes of* Q *at the meets of the* κ*-generator* (a_1x) *with* $H(a, b, c)$.

Finally, since

* For this last see (21) p. 13.

$$r = \frac{3}{(\mu\nu)^2} = \frac{1}{27}(mn)^6,$$

we have

(16) *To the quartic surface* $r = 0$ *in* S_3 *corresponds, by* T, Q *counting six times.*

This last theorem has a meaning only when we consider both T and its inverse. The indeterminateness of T is due partly to the explicit factor $3/(mn)^2$ and partly to the factor $(\sigma\tau) = -\sqrt{-r}$ employed in the definition of (λx), (μx), (νx). If we consider only the ratios of λ, μ, ν and of l, m, n we may say that

(17) *To every point on a tangent to* C^3 *at* $(\lambda_1 x)$ *corresponds by* T *the three points of* Q *in which the* κ*-generator* $(\lambda_1 x)$ *meets* a, b, c, *while to a point of* C^3 *corresponds no definite points in* Σ_3. *Inversely to every point of* Q *on the* κ*-generator* $(\lambda_1 x)$ *corresponds by* T^{-1} *the point* $(\lambda_1 x)$ *of* C^3 *except that to a point on* a, b, *or* c *corresponds no definite point of* S^3.

In connection with the general theorems (5) and (6) of § 3 we have the following:

(18) *A surface of order* n *in* S_3 *which contains* C^3 *as an* m*-tuple* p*-tuply asymptotic curve is transformed by* T *into a surface in* Σ_3 *of degree* $\nu = 3n - 4m - 2p$, *the quadric* Q *which appears* $2m + p$ *times being disregarded.*

For, according to (6) § 3, the most general surface of the above sort can be expressed in terms of the special surfaces which occur as the coefficients of $(px)^3$, $(\delta x)^2$, $(qx)^3$ and r. And there is at least one (and in fact only one) term homogeneous of degree n_2 in the coefficients of $(\delta x)^2$, n_3 in those of $(qx)^3$ and n_4 in r, and such that $n_2 + n_3 + 2n_4 = m$ and $n_3 + 2n_4 = p$. From (13), (15) and (16), Q separates out to a degree $2n_2 + 3n_3 + 6n_4 = 2m + p$ for this particular term and to a higher degree for the other terms.

Curves in S_3 are transformed by T into curves in Σ_3 which admit through every point at least one triple secant for they are made up of 6-points. We will consider only the lines of S_3. A line in general position, the intersection of two planes, is transformed by T into the intersection of two cubic surfaces of the system (11). Hence

(19) *To a line in general position in* S_3 *there correspond by* T *a curve of the sixth order which meets* Q *in the twelve points common to* a, b, c *and the four* κ*-generators defined by the tangents of* C^3 *met by the original line. Through every point of the curve passes a triple secant whose conjugate line as to* Q *is another triple secant.* a, b, c *are quadri-secants of the curve. If the line in* S_3 *is a tangent line of* $r = 0$ *in general position two of the four* κ*-generators coincide.*

Since a tangent of C^3 at (ax) is the intersection of the planes $(pa)^2(pb) = 0$ and $(pa)^3 = 0$ we have by taking the meet of the two corresponding cubic surfaces in Σ_3,

(20) *To a tangent of C^3 at (ax) corresponds in Σ_3 the κ-generator (ax), counting six times.*

A chord of C^3 is the pencil of cubics $(p_1x)^3 + \lambda(q_1x)^3$ and it meets C^3 at (σ_1x) and (τ_1x). From the properties of the 6-point it follows that

(21) *To a chord of C^3 corresponds in Σ_3 the two lines of a 6-point which are diagonals of the quadrilateral formed by $H(a, b, c)$ and the κ-generators (σ_1x) and (τ_1x) and further these two κ-generators each counting twice.*

A secant of C^3 at (b_1x) may be taken as part intersection of the quadric $(\delta a)^2 = 0$ and the plane $(pb_1)(pa_1)^2 = 0$, where (a_1x) is a factor of $(ax)^2$. To the plane corresponds in Σ_3 a cubic surface with the κ-generator (a_1x) as a double line, while to $(\delta a)^2 = 0$ corresponds a quadric with this generator as a single line. The remaining intersection due to the secant at (b_1x) is a curve of the fourth order and second kind. In the general curve of this type it happens four times that triple secants become tangent secants. But triple secants arising from 6-points cannot so degenerate and in fact the four tangent secants are replaced by two flex tangents. For let the secant of C^3 be given as the intersection of the two planes $(pb_1)(p\lambda_1)(p\mu_1) = 0$ and $(pb_1)(p\lambda_2)(p\mu_2) = 0$. We verify easily that the corresponding cubic surfaces in Σ_3 touch along the κ-generator. The remaining meet, a curve of fourth order, meets Q in eight points, six of which correspond to the two intersections of the original chord with $r = 0$ and lie on a, b, c. The other two are the meets of the κ-generator (b_1x) with $H(a, b, c)$. That they are flexes we may deduce from the following limit considerations. As the variable point on the chord of C^3 approaches (b_1x), its Hessian tends to a limiting value whose factors are (b_1x) and the polar of (b_1x) as to $(cx)^2$, the pair apolar to both $(\lambda_1x)(\mu_1x)$ and $(\lambda_2x)(\mu_2x)$. Three of the 6-point cluster around the one point where the κ-generator (b_1x) meets $H(a, b, c)$, the other three about the other point, each three however always lying on a diagonal of the Hessian quadrilateral. In the limit the two sets of three points coincide at the meets of (b_1x) with $H(a, b, c)$; the two lines of the 6-point become flex tangents and have for limiting positions the diagonals of the the quadrilateral form by $H(a, b, c)$ and the two κ-generators (b_1x) and $(cb_1)(cx)$.

To a chord of C^3 defined by the planes

$$(pb_1)(p\lambda_1)(p\mu_1) = 0 \qquad \textit{and} \qquad (pb_1)(p\lambda_2)(p\mu_2) = 0$$

corresponds in Σ_3, besides the κ-generator (b_1x) counting twice, a curve of the fourth order and second kind with two inflexions. The flex points are the intersections of (b_1x) with $H(a, b, c)$. The flex tangents are the diagonals of the quadrilateral on Q formed by $H(a, b, c)$ with the two κ-generators (b_1x) and the polar of (b_1x) as to the pair apolar to both $(\lambda_1x)(\mu_1x)$ and $(\lambda_2x)(\mu_2x)$.

§ 7. *The transforms by T^{-1} of manifolds in Σ_3.*

For the sake of brevity we shall content ourselves with an examination of the effect of T^{-1} upon the simpler manifolds, the planes and lines, of Σ_3. The results obtained will be sufficient to exhibit some of the properties of the algebraic group F_3.

In general it may be said that a manifold of order n, M^n, in Σ_3 must be considered in connection with the others obtained by replacing each point by the 6-point to which it belongs. The 6-M^n so obtained rather than the original M^n alone is transformed by T^{-1} into a manifold, M, in S_3. If the order of M be m, a line cuts it in m points in general position. In Σ_3 then a sextic curve cuts the 6-M^n in $6m$ points; whence $6m = 36n$.

(1) *T^{-1} carries an M^n (or also its 6-M^n) into an M^{6n} in S_3.*

We should expect then a plane of Σ_3 to be transformed by T^{-1} into a sextic surface. For convenience, however, we consider the effect of T upon the sextic

$$rr_1\,[(pp_1)^3]^2 - [(qq_1)^3]^2 = 0. \tag{2}$$

From the equations of T, $r = (mn)^6/27$ and from (5), § 6,

$$(pp_1)^3 = \frac{1}{27\sqrt{-r_1}}\,[\alpha^3 + \beta^3 - \gamma^3 - \delta^3],$$

where α, β, γ, δ are the linear expressions occurring in (5) in the order there written. Hence

$$rr_1\,[(pp_1)^3]^2 = -\frac{(mn)^6}{27^3}\,[\alpha^3 + \beta^3 - \gamma^3 - \delta^3]^2. \tag{3}$$

From (12) and (13) of § 6, we have

$$\begin{aligned}(qq_1)^3 &= \frac{(mn)^3}{27}\,[\{(mq_1) - (nq_1)\}\,\{(nq_1) - (lq_1)\}\,\{(lq_1) - (mq_1)\}]\\ &= \frac{(mn)^3}{27}\,[(l'q_1)(m'q_1)(n'q_1)]\\ &= \frac{(mn)^3}{27}\,[(l'\sigma_1)(m'\sigma_1)(n'\sigma_1) + (l'\tau_1)(m'\tau_1)(n'\tau_1)],\end{aligned}$$

since

$$(\sigma_1 x)^3 + (\tau_1 x)^3 = (q_1 x)^3.$$

By a change in the sign of (τx) in (5) of § 6, we have at once for this case

$$(qq_1)^3 = \frac{(mn)^3}{27^2}\,[\alpha'^3 + \beta'^3 + \gamma'^3 + \delta'^3],$$

in which α' is α written in primed variables, etc. Therefore

$$[(qq_1)^3]^2 = \frac{(mn)^6}{27^4}\,[\alpha'^3 + \beta'^3 + \gamma'^3 + \delta'^3]^2,$$

since

$$\alpha' = (\omega^2 - \omega)\alpha, \qquad \beta' = (\omega - \omega^2)\beta,$$

$$\gamma' = (\omega^2 - \omega)\gamma, \qquad \delta' = (\omega - \omega^2)\delta,$$

$$(4) \qquad [(qq_1)^3]^2 = -\frac{(mn)^6}{27^3}[\alpha^3 - \beta^3 + \gamma^3 - \delta^3]^2.$$

combining (4) with (3) we have

$$-rr_1[(pp_1)^3]^2 + [(qq_1)^3]^2 = \frac{(mn)^6}{27^3}\{[\alpha^3+\beta^3-\gamma^3-\delta^3]^2-[\alpha^3-\beta^3+\gamma^3-\delta^3]^2\}$$

$$= \frac{4}{27^3}(mn)^6(\alpha^3 - \delta^3)(\beta^3 - \gamma^3)$$

$$= \tfrac{4}{27}(mn)^6\{(\alpha-\delta)(\omega\alpha-\omega^2\delta)(\omega^2\alpha-\omega\delta)(\beta-\gamma)(\omega\beta-\omega^2\gamma)(\omega^2\beta-\omega\gamma)\}.$$

Since, to within a permutation which does not affect the result, we may write

$$(\lambda_1 x) = \frac{(\sigma_1 x) - (\tau_1 x)}{(\sigma_1 \tau_1)}, \qquad (\mu_1 x) = \frac{\omega(\sigma_1 x) - \omega^2(\tau_1 x)}{(\sigma_1 \tau_1)},$$

$$(\nu_1 x) = \frac{\omega^2(\sigma_1 x) - \omega(\tau_1 x)}{(\sigma_1 \tau_1)},$$

then

$$(\alpha - \delta) = \frac{1}{(\sigma_1 \tau_1)}[(l\lambda_1) + (m\mu_1) + (n\nu_1)],$$

$$(\omega\alpha - \omega^2\delta) = \frac{1}{(\sigma_1 \tau_1)}[(l\mu_1) + (m\nu_1) + (n\lambda_1)],$$

and so on for all the six planes of the 6-plane $\lambda_1\mu_1\nu_1$. We shall write the product of all six as $E_6(\lambda_1\mu_1\nu_1)$. Finally since $(\sigma_1\tau_1) = -\sqrt{-r_1}$, we have

$$(5) \qquad rr_1[(pp_1)^3]^2 - [(qq_1)^3]^2 = \frac{4}{27^3}\frac{(mn)^6}{r_1^3}E_6(\lambda_1\mu_1\nu_1).$$

The left side of this relation is the desired correspondent by T^{-1} of the $E_6(\lambda_1\mu_1\nu_1)$. For symmetry, however, we write it in a different form by the use of the reciprocity between the forms p_1 and q_1/r_1 given by STUDY,* i. e., we replace p_1 by q_1/r_1, q_1 by $-p_1/r_1$, r_1 by $1/r_1$, $(\lambda_1 x)$ by

$$(\lambda_1' x) = \frac{(\sigma_1 \tau_1)}{-\sqrt{-3}}\{(\mu_1 x) - (\nu_1 x)\}$$

and so on for $(\mu_1 x)$ and $(\nu_1 x)$. Equation (5) then takes the form

$$(6) \quad r[(pq_1)^3]^2 - r_1[(qp_1)^3]^2 = \frac{4}{27^4}r_1^9(mn)^6 \cdot E_6(\mu_1 - \nu_1, \nu_1 - \lambda_1, \lambda_1 - \mu_1).$$

* Loc. cit., p. 190.

Disregarding the extraneous factors, we have then

(7) *The 6-plane which is the polar as to Q of the 6-point $(\lambda_1 \mu_1 \nu_1)$, not on Q, is transformed by T^{-1} into the sextic surface in S_3 described in* (7), § 3, *namely*

$$S^6(p_1) \equiv r\,[(pq_1)^3]^2 - r_1\,[(qp_1)^3]^2.$$

Or if we call two 6-points each of which lies in the polar 6-plane of the other "conjugate 6-points," then

(8) *The vanishing of $r_2\,[(p_2 q_1)^3]^2 - r_1\,[(p_1 q_2)^3]^2$ is the condition that the 6-points in Σ_3 corresponding to the points $(p_1 x)^3$ and $(p_2 x)^3$ in S_3 be conjugate.*

A 6-plane in a special position is made up of tangent planes of Q through the same κ-generator $(t_1 x)$. One plane will have for equation

$$\rho(lt_1) + \sigma(mt_1) + \tau(nt_1) = 0, \qquad \rho + \sigma + \tau = 0,$$

the others being obtained from this by permuting ρ, σ, τ. There are four distinct cases:

(a) If $\rho : \sigma : \tau = 1 : \omega : \omega^2$, the 6-plane is the two planes determined by $H(a, b, c)$ with the κ-generator $(t_1 x)$ each counting three times.

(b) If $\rho : \sigma : \tau = 2 : 1 : 1$, the 6-plane is the three planes determined by a, b, c with $(t_1 x)$, each counting twice.

(c) If $\rho : \sigma : \tau = 0 : 1 : -1$, the 6-plane is the three planes determined by a', b', c' with $(t_1 x)$ each counting twice.

(d) The 6-plane is the six determined by the κ-generator $(t_1 x)$ with the six distinct h-generators obtained by permuting ρ, σ, and τ.

From (16), (10) and (14) of § 6 we have, for the first three cases,

(9) *The 6-plane of case (a) is transformed by T^{-1} into the cone $(\delta t_1)^2 = 0$ taken three times.*

(10) *The 6-plane of case (b) is transformed by T^{-1} into $(pt_1)^3 = 0$ taken twice ($r = 0$ being disregarded).*

(11) *The 6-plane of case (c) is transformed by T^{-1} into $(qt_1)^3 = 0$ taken twice.*

For case (d) we make use of the resolution * of the sextic $s^2 \cdot rp^2 + t^2 \cdot q^2 = 0$, where s and t are arbitrary parameters, into factors; one factor is

$$(\epsilon_1 R + \bar{\epsilon}_1 \bar{R})(\lambda x) + (\epsilon_2 R + \bar{\epsilon}_2 \bar{R})(\mu x) + (\epsilon_3 R + \bar{\epsilon}_3 \bar{R})(\nu x)$$

and the others are obtained by permuting the coefficients of (λx), (μx), (νx) in all possible ways. In this

$$R = \sqrt[3]{(s^2 - t^2)(s + t)} \qquad \text{and} \qquad \bar{R} = \sqrt[3]{(s^2 - t^2)(s - t)}.$$

Hence, replacing x by t_1, we have at once

(12) *The 6-plane of case (d) is transformed by T^{-1} into a member of the pencil of sextics*

* See STUDY, loc. cit., p. 191.

(13) $$s^2 \cdot r[(pt_1)^3] + t^2 \cdot [(qt_1)^3]^2.$$

For $s = t$, $s = 0$, $t = 0$ this gives cases (a), (b) and (c) respectively.

The doubly infinite system of sextics (13) is not contained in the triply infinite system $S^6(p_1)$ for the obvious reason that these systems as well as the system E_6 in Σ_3 are *non-linear*.

In order not to prolong the discussion unduly we consider only lines of Σ_3 in general position, i. e., having no particular situation with regard to Q; a, b, c; a', b', c'; or $H(a, b, c)$. Any line determines five others which form with it a 6-line. Two 6-points or two 6-planes determine however six 6-lines. To avoid this ambiguity we take the *one* line determined by two points $\rho_1(\lambda x)$, $\sigma_1(\mu x)$, $\tau_1(\nu x)$ and $\rho_2(\lambda x)$, $\sigma_2(\mu x)$, $\tau_2(\nu x)$ in which ρ_ι and λ_κ are symbols having an actual meaning only in the combinations $\rho_\iota \lambda_\kappa$, the convention for $\sigma_\iota \mu_\kappa$ and $\tau_\iota \nu_\kappa$ being the same. The line is then in parametral form $(\rho y)(\lambda x)$, $(\sigma y)(\mu x)$, $(\tau y)(\nu x)$ where $y_1 : y_2$ is the parameter and the identity in x and y

(14) $$(\rho y)(\lambda x) + (\sigma y)(\mu x) + (\tau y)(\nu x) = 0$$

holds. The corresponding locus in S_3 is

$$(\rho y)(\sigma y)(\tau y)(\lambda x)(\mu x)(\nu x),$$

i. e., a cubic curve which by reason of (14) osculates C^3 at the two points for which y is a root of $(\sigma y)(\tau y)(\mu \nu) = 0$. Hence

(15) *If a line in Σ_3 cuts Q on the two κ-generators $(\lambda_1 x)$ and $(\lambda_2 x)$, its transform by T^{-1} is a cubic space curve K^3 which osculates C^3 at the points $(\lambda_1 x)$ and $(\lambda_2 x)$.*

As a corollary from this we have

(16) *T^{-1} transforms a general manifold in Σ_3 of order n, M^n (or also the 6-M^n) into an M^{6n} in S_3 which contains C^3 as a $2n$-tuple $2n$-tuply asymptotic curve.*

For the 6-M^n is cut by a 6-line in $6n$ 6-points in general position. M^{6n} is cut by K^3 in $18n$ points only $6n$ of which can be in general position. The other $12n$ must be the six points of K^3 lying on C^3 each containing $2n$ times, i. e., M^{6n} has C^3 as a $2n$-tuple curve. If M^{6n} also contains C^3 as a p-tuply asymptotic curve it is transformed by T into a manifold of order $18n - 8n - 2p$ which must be $6n$, the order of the 6-M^n. Hence $p = 2n$.

A translation of some very obvious properties of points, 6-points and 6-planes gives rise to the following theorems:

(17) *Through two given points on C^3 and a given point $(p_1 x)^3$ passes one cubic curve K^3 which osculates C^3 at the given points.*

(18) *Through two given points $(p_1 x)^3$ and $(p_2 x)^3$ pass six curves K^3 which each osculate C^3 at two points.*

Such a set of six curves will be called a 6-$K^3(p_1 p_2)$.

(19) *Two sextic surfaces $S^6(p_1)$ and $S^6(p_2)$ intersect in six curves K^3 each of which osculates C^3 at two points.*

Such a set of six curves will be called a $6\text{-}\overline{K}^3(p_1, p_2)$. Of this latter set we can state the theorem

(20) *If $S^6(p_1)$ cuts the chord through p_2 in the points P_{11} and P_{12} and $S^6(p_2)$ cuts the chord through p_1 in the points P_{21} and P_{22}, the $6\text{-}\overline{K}^3(p_1 p_2)$ falls into two sets of three one set all passing through P_{11} and P_{21}, the other through P_{12} and P_{22}. The three pairs of points in which the three curves of one set osculates C^3 are in an involution whose double points are the pair apolar to $(\sigma_1 x)(\tau_2 x)$ and $(\sigma_2 x)(\tau_1 x)$; the involution of the other set has for double points the pair apolar to $(\sigma_1 x)(\sigma_2 x)$ and $(\tau_1 x)(\tau_2 x)$.*

For if

$$(l\lambda_1) + (m\mu_1) + (n\nu_1) = 0 \qquad \text{and} \qquad (l\lambda_2) + (m\mu_2) + (n\nu_2) = 0$$

fix the two 6-planes in Σ_3 corresponding to $S^6(p_1)$ and $S^6(p_2)$, their line of intersection cuts Q on the two κ-generators whose parameters are the factors of the quadratic

$$I \equiv \begin{vmatrix} (\lambda_1 x) & (\mu_1 x) & (\nu_1 x) \\ (\lambda_2 x) & (\mu_2 x) & (\nu_2 x) \\ 1 & 1 & 1 \end{vmatrix} = (\lambda_1 x)(\lambda_2' x) + (\mu_1 x)(\mu_2' x) + (\nu_1 x)(\nu_2' x) = 0$$

We obtain six lines corresponding to the $6\overline{K}^3$ by permuting *only* λ_2, μ_2, ν_2. The first part of the theorem follows from the property of the 6-plane lying on two lines. For the second we have, on adding the even permutations, I, II and III of I, that $I + II + III = 0$ and hence the three pairs of points are in an involution. From $I + \omega II + \omega^2 III$ we can factor out $(\sigma_2 x)$ leaving a factor $(\sigma_1 x)$. Similarly $I + \omega^2 II + \omega III$ factors into $(\tau_1 x)(\tau_2 x)$. Hence $(\sigma_1 x)(\sigma_2 x)$ and $(\tau_1 x)(\tau_2 x)$ are pairs of the involution. Also from the odd permutations we derive another involution containing the pairs $(\sigma_1 x)(\tau_2 x)$ and $(\sigma_2 x)(\tau_1 x)$.

From the duality between point and plane, line and line in Σ_3 we have in S_3 a duality between point p_1 and sextic $S^6_{(p_1)}$, between a $6\text{-}\overline{K}_3$ and a $6\text{-}K^3$. Since the quadratic I is the same for the parameters of the two κ-generators in which the line joining the *point* $(\lambda_1 x), \cdots$ and $(\lambda^2 x), \cdots$ cuts Q we have for the dual of (20)

(21) *If through the point p_1 and the chord through p_2 pass the two sextics $S^6(p_{11})$ and $S^6(p_{12})$; and if through p_2 and the chord through p_1 pass the two sextics $S^6(p_{21})$ and $S^6(p_{22})$, the $6\text{-}K^3$ through p_1 and p_2 is made up of two sets of three, one set lying on both $S^6(p_{11})$ and $S^9(p_{21})$; the other set on both $S^6(p_{12})$ and $S^6(p_{22})$. The six pairs of osculation points on C^3 lie in the same two invo-*

lutions described in (20), *the double points of the one involution being apolar to the double points of the other.*

As the correspondent of a pair of conjugate lines in Σ_3 we have

(22) *For every pair of points on a* K^3, p_1 *and* p_2, *the pair of sextics* $S^6(p_1)$ *and* $S^6(p_2)$ *have as part of their common curve a definite second* K^3 *which osculates* C^3 *at the same poiuts as the first. Two such* K^3 *which are moreover reciprocally related to each other will be called "conjugate."*

§ 8. *The algebraic group* F_3.

The equations of this group are given by (3), § 5, but we naturally prefer to obtain its properties from those of Φ_3 [(15), § 4] by means of T and T^{-1}. From § (6) and § (7) the translation is in most cases quite obvious. Unless definitely stated otherwise the following theorems refer to a general transformation of F_3, denoted by F *

(1) *Under the group* F_3, *there is a perfect duality between the point and sextic surface* $S^6(p_1)$ †; *between the* 6-K^3 *and the* 6-$\bar{K}^3$. *A pair of conjugate* K^3*'s are self-dual.*

(2) *A point,* p_1, *is transformed by* F *into six points which lie by threes on two conjugate* K^3*'s, each of which osculates* C^3 *at the two meets of* C^3 *with its chord through* p_1.

(3) *A sextic* $S^6(p_1)$ *is transformed by* F *into six such sextics which pass by threes through two conjugate* K^3*'s osculating* C^3 *at its two intersections with its chord through* p_1.

(4) *A* K^3 *through a point* p_1 *is transformed by* F *into six* K^3*'s, each of which osculates* C^3 *at the same points as the original* K^3 *and passes through one of the transforms of* p_1.

(5) *The system of sextics* $S^6(p_1)$ *and the system of cubic curves* K^3 *are the manifolds of lowest degree which are transformed among each other by the transformations of* F_3.

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* This transformation will be viewed in a different manner from that customary in the theory of LIE. From the equations of the group we see that the coordinates of the transformed point $(p'x)^3 = r'(\lambda' x)(\mu' x)(\nu' x)$ are *six-valued* functions of the coordinates of the original point. These algebraic functions have for branch points the entire surface $r = 0$. On every manifold, then, of dimensions greater than zero will lie some of these singular points. So that it seems — at any rate when manifolds are in question — simpler and more in accord with the nature of the group to consider the various branches of the algebraic functions *simultaneously*. This requires however an extended definition of a group. For if a point is transformed by F into six points, the *successive* performance of two transformations of the group is equivalent to the *simultaneous* performance of a finite number (in the present case generally six) of transformations of the group.

Or, using the word transformation in the ordinary sense, we may say that the transformations of Φ_3 fall into sets of six and such a set will be denoted by F.

† We assume of course that the points, curves, and surfaces considered in (1), (2), (3) and (4) are general, i. e. have no particular situation with regard to $r = 0$.

For any surface in S_3 is transformed by T into a surface of order $3n - 4m - 2p$ in Σ_3 which has however a particular situation with regard to the triple of generator a, b, c or its covariants. This special situation is destroyed by a transformation of Φ_3 and the transformed surface is carried again by T^{-1} into a surface in Σ_3 of degree greater than n. For example:

(6) *A plane in S_3 is transformed by F into a surface of order 18 which contains C^3 as a six-tuple six-tuply asymptotic curve with seven-tuple points at the intersections of the original plane with C^3.*

Hence the required manifolds of lowest degree in S_3 arise from general manifolds of lowest degree in Σ_3, namely the planes and lines.

In order to characterize more completely the six curves into which a K^3 is transformed by F, we may introduce the doubly infinite system of sextics, (13) § 7, any one of which will be denoted by Σ^6 or $\Sigma^6(t_1, s/t)$. On an $S^6(p_1)$ lies a doubly infinite system of K^3, each of which is characterized by its two points of osculation with C^3. On a $\Sigma^6(t_1 s/t)$ there is also a doubly infinite system of K^3, all of which osculate C^3 at $(t_1 x)$. Two Σ^6 intersect also in six K^3 and we have

(7) *A K^3 osculating C^3 at $(t_1 x)$ and $(t_2 x)$ is transformed by F into six curves K^3 which form the complete intersection of two definite sextics $\Sigma^6(t_1 s/t)$ and $\Sigma^6(t_2, s'/t')$.*

These results seem sufficient to demonstrate the value of the canonical forms employed for the various groups. For the sake of brevity no reference has been made to the transformation S as it appears in the space S_3, where, with G_3 and F_3, it generates a six-parameter algebraic group.

BONN, *August*, 1904.